Renormalization group transformation in models on a generalized hierarchical lattice

Missarov M.D.

Kazan Federal University

p-adics 2021, May 17-28 2021

Gaussian self-similar fields on the hierarchical lattice

Let $T = \{0, 1, 2, \ldots\}$ and let $V_k^s = \{j \in T : \; k \cdot 2^s \le j < (k+1) \cdot 2^s\}$, where $k \in T$, $s \in N = \{1, 2, 3, \ldots\}$. The hierachical distance $d_2(i,j)$, $i, j \in T$, $i \neq j$ is defined as $d_2(i,j) = 2^{s(i,j)}$, where

$$s(i,j) = \min\{s : \; \text{there is } k \in T \text{ such thet } i, j \in V_k^s\}.$$

Let $T^2 = T \times T$, $k = (k_1, k_2) \in T^2$,

$$V_k^s = \{(j_1, j_2) \in T^2 : \; k_1 \cdot 2^s \le j_1 < (k_1+1) \cdot 2^s, \; k_2 \cdot 2^s \le j_2 < (k_2+1) \cdot 2^s\}.$$

For any $k = (k_1, k_2) \in T^2$, $l = (l_1, l_2) \in T^2$, $k \neq l$ we define $s(k,l) = \max(s(k_1, l_1), s(k_2, l_2))$. Then hierachical distance on T^2 is generalized as $d_2(k,l) = 2^{s(k,l)}$.

Gaussian self-similar fields on the hierarchical lattice

Let us consider the following functions on T^2:

$$d(k,l;\lambda) = d_2(k,l), \text{ if } s(k_1,l_1) \neq s(k_2,l_2),$$

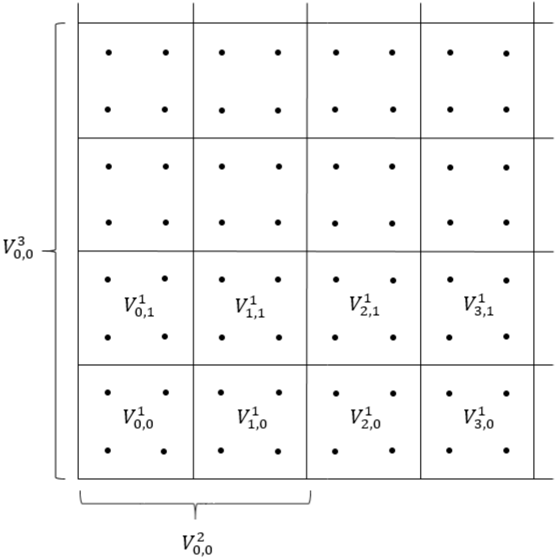
$$d(k,l;\lambda) = \lambda d_2(k,l), \text{ if } s(k_1,l_1) = s(k_2,l_2),$$

$$f(k,l;\lambda;\alpha) = d^\alpha(k,l;\lambda), \text{ if } k \neq l,$$

$$f(k,k;\lambda;\alpha) = \frac{2+\lambda^\alpha}{4(1-2^{-(2+\alpha)})},$$

λ is a real-valued parameter, $\lambda > 0$.

figure 1



Gaussian self-similar fields on the hierarchical lattice

Let ψ^* denote some bosonic or femionic field on the hierarchical lattice. The block-spin renormalization group transformation is defined by the formula

$$\psi^{*\prime}(i) \equiv (r(\alpha)\psi^*)(i) = 2^{-\alpha/2} \sum_{j \in V_i^1} \psi^*(j), \qquad (1)$$

where $\alpha \in R^1$ is the renormalization group parameter.
In the fermionic case we consider 4-component fermionic field

$$\psi^*(i) = (\bar{\psi}_1(i), \psi_1(i), \bar{\psi}_2(i), \psi_2(i)), \quad i \in T^2,$$

where the components are generators of a Grassman algebra.

Gaussian self-similar fields on the hierarchical lattice

Let us redenote V_0^N by Λ_N. Let Γ_N be the Grassmann subalgebra generated by $4 \cdot 2^N$ generators $\bar{\psi}_1(i), \psi_1(i), \bar{\psi}_2(i), \psi_2(i),\ i \in \Lambda_N$. The Gaussian fermionic field with zero mean and binary correlation function

$$\langle \psi_n(k)\bar{\psi}_m(l)\rangle = \delta_{n,m} b(k,l), \quad n,m = 1,2, \quad k,l \in T^2$$

is defined on the whole laftice as a quasistate (expectation value) on the algebra of all monomials such that $\langle F \rangle$ for an even degree monomial F is calculated using the Wick rule and $\langle F \rangle = 0$ for any odd degree monomial.
Let $b(k,l;\lambda;\alpha) = f(k,l;\lambda;\alpha - 4)$.

Gaussian self-similar fields on the hierarchical lattice

Lemma

The Gaussian fermionic field with zero mean and binary correlation function

$$\langle \psi_n(k)\bar{\psi}_m(l)\rangle = \delta_{n,m} b(k,l;\lambda;\alpha), \quad n,m = 1,2, \quad k,l \in T^2$$

is invariant under renormalization group transformation with parameter α *.*

Proof. Note, that if $i \in V_k^1$, $j \in V_l^1$, then $d(i,j) = 2d(k,l)$ and $s(k_1,l_1) = s(k_2,l_2)$ if and only if $s(i_1,j_1) = s(i_2,j_2)$.
If $k \neq l$, then

$$\langle \psi_1'(k)\bar{\psi}_1'(l)\rangle = 2^{-\alpha} \sum_{i\in V_k^1} \sum_{j\in V_l^1} \langle \psi_1(i)\bar{\psi}_1(j)\rangle =$$

$$= 2^{-\alpha} \sum_{i\in V_k^1} \sum_{j\in V_l^1} f(2k+i, 2l+j; \lambda; \alpha-4) =$$

$$= 2^{-\alpha} \cdot 2^{\alpha-4} \cdot 4^2 \cdot f(k,l,\lambda;\alpha-4) = b(k,l;\lambda;\alpha).$$

Gaussian self-similar fields on the hierarchical lattice

Let $k = l$. Note that

$$V_k^1 = \{(2k_1, 2k_2), (2k_1+1, 2k_2), (2k_1, 2k_2+1), (2k_1+1, 2k_2+1)\}.$$

It follows that

$$\langle \psi_1'(k) \bar{\psi}_1'(k) \rangle = 2^{-\alpha} \sum_{i \in V_k^1} \sum_{j \in V_k^1} \langle \psi_1(i) \bar{\psi}_1(j) \rangle =$$

$$= 2^{-\alpha} (b(k,k;\lambda;\alpha) + 2^{\alpha-4} \cdot 2 + \lambda^{\alpha-4} \cdot 2^{\alpha-4}) \cdot 4 = b(k,k;\lambda;\alpha).$$

The same is true for all other correlations. The lemma is proved. To study non - Gaussian states we use the Gibbs description of the field. Let us consider the restriction of the Gaussian field ψ^* on the volume Λ_N. Let us denote the matrix of correlations of this restriction as $B_N(\lambda;\alpha) = (b(k,l;\lambda;\alpha))_{k,l \in \Lambda_N}$.

Gaussian self-similar fields on the hierarchical lattice

To describe this restriction in Gibbs form, we must invert the matrix $B_N(\lambda;\alpha)$.
Let $H_N(\lambda,\mu;\alpha) = (h_N(k,l;\lambda,\mu;\alpha))_{k,l\in\Lambda_N}$, where

$$h_N(k,l;\lambda,\mu;\alpha) = g(\lambda,\mu;\alpha)h_0(k,l;\mu;\alpha) - C(N;\lambda,\mu;\alpha), \tag{2}$$

$$h_0(k,l;\mu;\alpha) = f(k,l;\mu;-\alpha), \tag{3}$$

$g(\lambda,\mu;\alpha)$ and $C(N;\lambda,\mu;\alpha)$ are some normalizing functions:

$$g(\lambda,\mu;\alpha) = \left(\frac{(2+\mu^{-\alpha})(2+\lambda^{\alpha-4})}{16(2^{2-\alpha}-1)(2^{\alpha-2}-1)} + \frac{2+\lambda^{\alpha-4}\mu^{-\alpha}}{12}\right)^{-1}, \tag{4}$$

$$C(N;\lambda,\mu;\alpha) = -g(\lambda,\mu;\alpha)\cdot\frac{(2+\lambda^{\alpha-4}\mu^{-\alpha})(2^{\alpha-2}-1)}{12(2+\lambda^{\alpha-4})}\cdot 2^{-\alpha(N+1)}. \tag{5}$$

Gaussian self-similar fields on the hierarchical lattice

Theorem

If $\alpha > 2,\ \alpha \neq 4$, then for all λ such that

$$\left(\frac{3 \cdot 2^{2-\alpha}}{1 - 2^{2-\alpha}}\right)^{\frac{1}{\alpha-4}} < \lambda < \left(\frac{4(1 - 2^{2-\alpha})}{3}\right)^{\frac{1}{\alpha-4}} \tag{6}$$

the matix $H_N(\lambda, \mu(\lambda); \alpha)$ is the inverse of the matrix $B_N(\lambda; \alpha)$, where

$$\mu(\lambda) = \left(\frac{4(1 - 2^{2-\alpha}) - 3\lambda^{\alpha-4}}{\lambda^{\alpha-4}(1 - 2^{2-\alpha}) - 3 \cdot 2^{2-\alpha}}\right)^{-\frac{1}{\alpha}}. \tag{7}$$

Proof see in [1].
One can show that $\mu(\lambda) = \lambda$ for positive λ if and only if $\lambda = 1, 2^{-1/2}, 2^{-1}$.

Renormalization group transformation in the space of Hamiltonians

Let us fix some value of λ, satisfying to the condition (6), let $\mu(\lambda)$ is given by (7), and let us simplify notations:

$$a = g(\lambda, \mu(\lambda); \alpha), \quad C(N) = C(N; \lambda, \mu(\lambda); \alpha),$$

$$h_{0,N}(k, l) = af(k, l; \mu(\lambda); -\alpha) - C(N).$$

Let us consider the Gaussian Hamiltonian

$$H_{0,N}(\psi^*; \lambda; \alpha) = \sum_{k,l \in \Lambda_N} h_{0,N}(k, l) \bar{\psi}(k) \psi(l).$$

and the restriction of the Gaussian state $\rho_0(\alpha)$ on the volume Λ_N.

Renormalization group transformation in the space of Hamiltonians

From the theorem 1 it follows, that for any $F(\psi^*) \in \Gamma_N$

$$\rho_0(\alpha)(F(\psi^*)) = Z_N^{-1}(\alpha) \int F(\psi^*) \exp\{-H_{0,N}(\psi^*; \alpha)\} \, d\psi^*,$$

where the superanalysis integration rule is defined as

$$d\psi^* = \prod_{i \in \Lambda_N} d\psi_1(i)\, d\bar{\psi}_1(i)\, d\psi_2(i)\, d\bar{\psi}_2(i),$$

$$\int \psi_k(i)\, d\psi_k(i) = \int \bar{\psi}_k(i)\, d\bar{\psi}_k(i) = 1, \int d\psi_k(i) = \int d\bar{\psi}_k(i) = 0, \quad (8)$$

$$H_{0,N}(\psi^*; \alpha) = \sum_{i,j \in \Lambda_N} h_{0,N}(i,j) \bar{\psi}(i) \psi(j), Z_N(\alpha) = \int \exp\{-H_{0,N}(\psi^*; \alpha)\} \, d\psi^*$$

Renormalization group transformation in the space of Hamiltonians

Let us consider local potential (self-interaction) for the 4-component fermionic field

$$L(\psi^*; r, g) = r(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + g\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2.$$

Let us denote $u(\psi^*(i)) = \exp\{-L(\psi^*(i); r, g)\}$. We will use the notation $\rho_N(\alpha; u)$ for the state on algebra Γ_N, given by the formula

$$\rho_N(\alpha; u)(F) = Z_N^{-1}(\alpha; u)\langle F \exp\{-H_{0,N}(\psi^*; \alpha)\} \prod_{i\in\Lambda_N} u(\psi^*(i))\rangle_0,$$

$$Z_N(\alpha; u) = \langle \exp\{-H_{0,N}(\psi^*; \alpha)\} \prod_{i\in\Lambda_N} u(\psi^*(i))\rangle_0,$$

where $\langle \cdot \rangle_0$ denote integral on the Γ_N (with rules (8)), If ρ is a state on Γ_N, then the renormalized state ρ' is defined on Γ_{N-1} by

$$\rho'\left(F(\psi^{*\prime})\right) = \rho\left(F(r(\alpha)\psi^*)\right).$$

Renormalization group transformation in the space of Hamiltonians

We shall use the following notations:

$$\psi(i) = (\psi_1(i), \psi_2(i)), \quad \bar{\psi}(i) = (\bar{\psi}_1(i), \bar{\psi}_2(i)),$$

$$\bar{\psi}(i)\eta(i) = \bar{\psi}_1(i)\eta_1(i) + \bar{\psi}_2(i)\eta_2(i), \quad i \in T^2.$$

Let us define the quadratic form

$$a \sum_{i,j \in V_k^1} f(i,j;\mu(\lambda);-\alpha)\bar{\eta}(i)\eta(j) = Q_k(\eta^*), \quad k \in \Lambda_{N-1}.$$

Recall, that for all $i, j \in V_k^1$

$$f(i,j;\mu;-\alpha) = \frac{2+\mu^{-\alpha}}{4(1-2^{\alpha-2})}, \quad \text{if } i = j,$$

$$f(i,j;\mu;-\alpha) = 2^{-\alpha}, \quad \text{if } d_2(i,j) = 1 \text{ and } s(i_1,j_1) \neq s(i_2,j_2),$$

$$f(i,j;\mu;-\alpha) = (2\mu)^{-\alpha}, \quad \text{if } d_2(i,j) = 1 \text{ and } s(i_1,j_1) = s(i_2,j_2).$$

Renormalization group transformation in the space of Hamiltonians

Theorem

Let $Z_N(\alpha; u) \neq 0$. *Then*

$$\rho'_N(\alpha; u) = \rho_{N-1}(\alpha; u'),$$

where

$$u'(\psi^{*\prime}) =$$

$$= \int \delta\left(\sum_{i \in V_0^1} \eta^*(i)\right) \exp\{-Q_0(\eta^*)\} \prod_{i \in V_0^1} \left(u(2^{\alpha/2-2}\psi^{*\prime} + \eta^*(i)\right) \, d\eta^*(i). \tag{9}$$

Proof. Let $\psi^{*\prime} = r(\alpha)\psi^*$ is given by (1). Consider integral

$$I = \langle F(\psi^{*\prime}) \exp\{-H_{0,N}(\psi^*; \alpha)\} \prod_{i \in \Lambda_N} u(\psi^*(i))\rangle_0.$$

Renormalization group transformation in the space of Hamiltonians

We introduce new variables
$\eta^*(i) = (\bar\eta_1(i), \eta_1(i), \bar\eta_2(i), \eta_2(i))$, $i \in \Lambda_N$:

$$\psi^*(i) = 2^{\alpha/2-2}\psi^{*\prime}(k) + \eta^*(i), \quad i \in V_k^1, \quad k \in \Lambda_{N-1}. \tag{10}$$

Note that

$$\sum_{i\in V_k^1} \eta^*(i) = 0, \quad k \in \Lambda_{N-1}.$$

Then

$$H_{0,N}(\psi^*;\alpha) =$$

$$= \sum_{k,l\in\Lambda_{N-1}} \sum_{i\in V_k^1} \sum_{j\in V_l^1} h_{0,N}(i,j) \left(2^{\alpha/2-2}\bar\psi'(k) + \bar\eta(i)\right) (2^{\alpha/2-2}\psi'(l) + \eta(j) =$$

$$= \sum_{k,l\in\Lambda_{N-1}} \bar\psi'(k)\psi'(l) \cdot 2^{\alpha-4} \sum_{i\in V_k^1} \sum_{j\in V_l^1} h_{0,N}(i,j) +$$

$$+ \sum_{k\in\Lambda_{N-1}} \sum_{i,j\in V_k^1} h_{0,N}(i,j)\bar\eta(i)\eta(j). \tag{11}$$

Renormalization group transformation in the space of Hamiltonians

The expansion (11) follows from the two facts.First,

$$\sum_{k,l\in\Lambda_{N-1}}\sum_{i\in V_k^1}\sum_{j\in V_l^1} h_{0,N}(i,j)\cdot 2^{\alpha/2-2}\psi'(k)\eta(j) =$$

$$= \sum_{k,l\in\Lambda_{N-1}} 2^{\alpha/2-2}\psi'(k)\sum_{j\in V_l^1}\sum_{i\in V_k^1} h_{0,N}(i,j)\eta(j).$$

As $\sum_{i\in V_k^1} h_{0,N}(i,j) = c_1(k,l)$ for all $j\in V_l^1$, where $c_1(k,l)$ is some constant depending only on k and l. Then

$$\sum_{j\in V_l^1}\sum_{i\in V_k^1} h_{0,N}(i,j)\eta(j) = \sum_{j\in V_l^1} c_1(k,l)\sum_{i\in V_k^1}\eta(i) = 0.$$

Therefore we have no terms of the type $\psi'(k)\bar{\eta}(j)$ or $\bar{\psi}'(k)\eta(j)$ in the expansion (11).

Renormalization group transformation in the space of Hamiltonians

Next, let us assume that $k \neq l$. Then $h_{0,N}(i,j) = c_2(k,l)$, where $c_2(k,l)$ is some constant depending only on k and l. Therefore,

$$\sum_{i\in V_k^1} \sum_{j\in V_l^1} h_{0,N}(i,j)\bar{\eta}(i)\eta(j) = c_2(k,l) \left(\sum_{i\in V_k^1} \bar{\eta}(i) \right) \left(\sum_{j\in V_l^1} \eta(j) \right) = 0.$$

As $\psi^*(k)$, $k \in \Lambda_{N-1}$ is renormalization group transformation of the field $\psi^*(i)$, $i \in \Lambda_N$, which is invariant relative to this transformation, then

$$H_{0,N}(\psi^*;\alpha) = H_{0,N-1}(\psi^{*\prime};\alpha) + \sum_{k\in\Lambda_{N-1}} \sum_{i\in V_{k,1}} h_{0,N}(i,j)\bar{\eta}(i)\eta(i).$$

Renormalization group transformation in the space of Hamiltonians

Let us denote a "Berezinian" of the linear change of variables (10) as $b(N, \alpha)$. Then

$$I = (b(n, N, \alpha))^{-1} \int F(\psi^{*\prime}) \exp\{-H_{0,N-1}(\psi^{*\prime}; \alpha)\} \times$$

$$\times \prod_{k \in \Lambda_{N-1}} \int \delta \left(\sum_{i \in V_k^1} \eta^*(i) \right) \exp\{-Q_k(\eta^*)\} \times$$

$$\times \prod_{i \in V_k^1} u(2^{\alpha/2-2} \psi^{*\prime}(k) + \eta^*(i)) \, d\eta^*(i) \prod_{k \in \Lambda_{N-1}} d\psi^{*\prime}(k).$$

Here $\delta(\eta^*)$ is a delta function defined by the condition

$$\int \delta(\eta^*) f(\eta^*) \, d\eta^* = f(0).$$

Theorem is proved.

Renormalization group transformation in the space of Hamiltonians

From (9) we see that renormalization group transformation is local and does not depend on the subvolume Λ_N. Therefore this formula is true for the model on the whole lattice T^2.
In the bosonic case the action of RG-transformation in the space of free density measures $f(\xi)$ is given by similar expession

$$f'(\xi') =$$

$$= \int \delta \left(\sum_{i \in V_0^1} \eta(i) \right) \exp\{-Q_0(\eta)\} \prod_{i \in V_0^1} \left(f(2^{\alpha/2-2}\xi' + \eta(i) \right) \, d\eta(i),$$

where

$$Q_0(\eta) = a \sum_{i,j \in V_0^1} f(i,j;\mu(\lambda);-\alpha)\eta(i)\eta(j).$$

Discretization of p-adic fields

As an example of bosonic theory we consider φ^4-theory with the action

$$H(\varphi; r, g) = H_0(\varphi; \alpha) + \int L(\varphi(x); r, g)\, dx, \tag{12}$$

where

$$L(\varphi(x); r, g) = r\varphi^2(x) + g\varphi^4(x).$$

The Gaussian part is

$$H_0(\varphi; \alpha) = \frac{c(\alpha)}{2} \int |x - y|^{-\alpha} \varphi(x)\varphi(y)\, dxdy, \tag{13}$$

while dx is the Lebesgue measure in the Euclidean case and Haar measure in p-adic case.

Discretization of p-adic fields

As an example of the fermionic theory we consider the four–component field $\psi^*(x) = (\overline{\psi}_1(x), \psi_1(x), \overline{\psi}_2(x), \psi_2(x))$ over Q_p^d, whose components are generators of tne Grassmann algebra. Let the Gibbs state describing this field be determined by the Hamiltonian

$$H(\psi^*; \alpha; r, g) = H_0(\psi^*; \alpha) + \int L(\psi^*(x); r, g)\, dx, \qquad (14)$$

$$H_0(\psi^*; \alpha) =$$

$$= c(\alpha) \int ||x - y||^{-\alpha} \left(\overline{\psi}_1(x)\psi_1(y) + \overline{\psi}_2(x)\psi_2(y) \right) dxdy.$$

The Lagrangian is

$$L(\psi^*(x); r, g) = r(\overline{\psi}_1(x)\psi_1(x) + \overline{\psi}_2(x)\psi_2(x)) + \\ + g(\overline{\psi}_1(x)\psi_1(x)\overline{\psi}_2(x)\psi_2(x)).$$

Discretization of p-adic fields

The Gaussian Hamiltonian is invariant w.r.t.the group of scaling transformations $(S_\lambda(\alpha)\varphi)(x) = |\lambda|^{d-\alpha}\varphi(\lambda x)$, where $\alpha \in R$ is the parameter of this group.
The value $\alpha = d + 2$ corresponds to the Laplace operator in the Euclidean case or its p-adic analog.
Every p-adic number is reprsented in the form

$$x = \sum_{i=-n(x)}^{\infty} c_i p^i, \quad c_i \in \{0, 1, \dots, p-1\}$$

$$x = \underbrace{\dots c_3 c_2 c_1 c_0}_{\text{integer part}}, \underbrace{c_{-1} \dots c_{-n(x)}}_{\text{fractional part}}, \; |x|_p = p^{n(x)}.$$

$\{x\} = c_{-n}p^{-n} + \ldots + c_{-1}p^{-1}$ is fractional part of x.
For $x = (x_1, \dots, x_d) \in Q_p^d$ we put
$||x||_p = \max_i ||x_i||_p, \; \{x\} = (\{x_1\}, \dots, \{x_d\})$.

Discretization of p-adic fields

The discrete set $T_p^d = \left\{x \in Q_p^d : x = \{x\}\right\}$ — hierarchical lattice with the elementary cell size $n = p^d$ and with the hierarchical distance $d(i,j) = |i-j|_p,\ i,j \in T_p^d$. Let $Z_p^d = \left\{x \in Q_p^d : ||x||_p \leq 1\right\}$

$$Q_p^d = \bigcup_{j \in T_p^d} \{j + Z_p^d\}.$$

The discretization of the field φ is the field ξ over T_p^d such that

$$\xi(j) = \int \varphi(j+x)\chi(x)\, dx, \quad j \in T_p^d.$$

The discretization ζ of the field $(S_{p^{-1}}(\alpha)\varphi)(x)$ is the transformation of the hierarchical block-spin Kadanoff-Wilson renormalization group over the field ξ,

$$\zeta(j) = (r(\alpha)\xi)(j) = p^{-\alpha/2} \sum_{i \in B(j)} \xi(i),$$

where $B(j) = \{i \in T_p^d : ||i - p^{-1}j|| \leq p\}$ are the elementary blocks in the hierarchical lattice.

Discretization of p-adic fields

The discretization of the Gaussian field φ with the Hamiltonian (12) is a Gaussian field $\xi(j)$ with the Hamiltonian

$$H_0'(\xi;\alpha) = 1/2 \sum_{i,j} h(i,j;\alpha)\xi(i)\xi(j), \tag{15}$$

$$h(i,j;\alpha) = \frac{f_p(\alpha)}{f_p(d-\alpha)} \left(1-\delta_{i,j}\right) ||i-j||^{-\alpha} + \frac{f_p(\alpha)}{f_p(d)} \delta_{i,j}.$$

Discretization of p-adic fields

Now we consider the four–component field $\psi^*(x) = (\overline{\psi}_1(x), \psi_1(x), \overline{\psi}_2(x), \psi_2(x))$ over Q_p^d, whose components are generators of tne Grassmann algebra. Let the Gibbs state describing this field be determined by the Hamiltonian

$$H(\psi^*; \alpha; r, g) = H_0(\psi^*; \alpha) + \int L(\psi^*(x); r, g)\, dx, \qquad (16)$$

whose Gaussian part is

$$H_0(\psi^*; \alpha) = c(\alpha) \int ||x - y||^{-\alpha} \left(\overline{\psi}_1(x)\psi_1(y) + \overline{\psi}_2(x)\psi_2(y) \right) dxdy.$$

The Lagrangian is

$$L(\psi^*(x); u, v) = u(\overline{\psi}_1(x)\psi_1(x) + \overline{\psi}_2(x)\psi_2(x)) + \\ + v(\overline{\psi}_1(x)\psi_1(x)\overline{\psi}_2(x)\psi_2(x)).$$

Discretization of p-adic fields

Let

$$\xi^*(j) = \int \psi^*(j + x)\chi(x)\, dx, \quad j \in T_p^d.$$

The Hamiltonian of the field ξ^* is

$$H'(\xi^*; \alpha; u, v) = H'_0(\xi^*; \alpha) -$$

$$- \ln \sum_j < \exp\{- \int L(\xi^*(j) + \eta^*(x); u, v)\, dx\} >_{\mu(d\eta^*)},$$

where the average is taken w.r.t. the measure $\mu(d\eta^*)$, which corresponds to the Gaussian field $\eta^*(x)$ and has the support belonging to Z_p^d, a zero mean value, and the binary correlation function

$$< \overline{\eta}_k(x)\eta_l(y) >= \delta_{kl} \left(\frac{f_p(2d-\alpha)}{f_p(\alpha-d)} ||x-y||^{(\alpha-2d)} - \frac{f_p(2d-\alpha)}{f_p(d)} \right),$$

$$< \overline{\eta}_k(x)\overline{\eta}_l(y) >=< \eta_k(x)\eta_l(y) >= 0,$$

$$H'_0(\xi^*; \alpha) = \sum_{i,j} h(i, j; \alpha)\ \left(\overline{\xi}_1(i)\xi_1(j) + \overline{\xi}_2(i)\xi_2(j)\right).$$

Discretization of p-adic fields

Thus, the discretization of the field with Hamiltonian (14) leads to the hierarchical model Hamiltonian with the potential $L(\xi^*; r(u,v), g(u,v))$, where the coupling constants r and g of the lattice field can be determined from the nontrivial functional integral

$$< \exp\{-\int L(\xi^* + \eta^*(x); u, v) dx\} >_{\mu(d\eta)} .$$

Discretization of p-adic fields

Let $x = (x_1, x_2) \in Q_2^2$. We define the quantity $|x| = \sqrt{|x_1^2 + x_2^2|_2}$, where $|\cdot|_2$ denotes the 2-adic norm.

Lemma

If $x = (x_1, x_2),\ x_i \in Q_2, i = 1, 2, \quad |x_1|_2 \neq |x_2|_2$, *then* $|x| = \max(|x_1|_2, |x_2|_2)$, *and if* $|x_1|_2 = |x_2|_2$, *then* $|x| = 2^{-1/2}|x_1|_2$. *The quantity* $|x|$ *is a norm on 2-adic space.*

If we will use instead of norm $|x|_2 = \max_i |x_i|_2$ the norm $|x| = \sqrt{|x_1^2 + x_2^2|_2}$, then after discretization we obtain hierarchical model with distance $d(i, j; 2^{-1/2})$ (see [2]).

ε-expansions in p-adic models

Let ξ be a field on the lattice T_p^d. The hierarchical renormalization block-spin transformation is given as

$$r(\alpha)\xi(j) = p^{-\alpha/2} \sum_{l \in T_p^d : |pl - j|_p \leq p} \xi(l),$$

α is real-valued renormalization group parameter.
We define field $\sigma(k)$ in the unit ball of d-dimensional p-adic space $\{k : |k|_p \leq 1\}$ by Fourier series

$$\sigma(k) = \sum_{j \in T_p^d} \exp\{-2\pi i(j, k)\}\xi(j),$$

where $(j, k) = j_1 k_1 + \ldots + j_d k_d$. It is easy to find the action of RG - transformation $r(\alpha)$ in terms of the field σ:
$r(\alpha)\sigma(k) = p^{-\alpha/2}\sigma(kp)\chi(k)$. Here $\chi(k)$ denotes characteristic function of the unit ball.

ε-expansions in p-adic models

From this point we shall describe the hierarchical and Euclidean models in the similar way. Let $\Omega = \{k : |k| \leq 1\}$ denotes the unit ball in the d-dimensional Euclidean or p-adic space Q_p^d (we'll omit index p in the notation of p-adic norm).

Let $\sigma(k)$ be a field defined on this ball, which will denote in this paper bosonic (complex-valued) field or 4-component fermionic field $\sigma(k) = (\overline{\sigma_1}(k), \sigma_1(k), \overline{\sigma_2}(k), \sigma_2(k))$, where the components are generators of the Grassmann algebra.The Wilson's renormalization group transformation (RG) in terms of realizations of the field $\sigma(k)$ is defined as

$$r_\lambda(\alpha)\sigma(k) = |\lambda|^{-\alpha/2}\sigma(k/\lambda)\chi(k),$$

where λ is real or p-adic number, $\chi(k)$ is the characteristic function of the ball Ω.

Then Gaussian fields with binary correlation function

$$\langle\sigma(k_1)\sigma(k_2)\rangle = \delta(k_1 + k_2)|k_1|^{d-\alpha}\chi(k_1)$$

in the bosonic case and

ε-expansions in p-adic models

and

$$\langle \sigma_i(k_1)\overline{\sigma_j}(k_2)\rangle = \delta(k_1+k_2)\delta_{ij}|k_1|^{d-\alpha}\chi(k_1),$$
$$\langle \overline{\sigma_i}(k_1)\overline{\sigma_j}(k_2)\rangle = \langle \sigma_i(k_1)\sigma_j(k_2)\rangle = 0, \quad i,j=1,2$$

in the fermionic one are invariant under transformation $r_\lambda(\alpha)$. In the Gibbsian form these Gaussian fields are described by the Hamiltonians

$$H_0(\sigma;\alpha) = \frac{1}{2}\int\limits_{\Omega^2} \delta(k_1+k_2)|k_1|^{\alpha-d}\sigma(k_1)\sigma(k_2)\,dk_1dk_2$$

in the bosonic case and

$$H_0(\sigma;\alpha) = \int\limits_{\Omega^2} \delta(k_1+k_2)|k_1|^{\alpha-d}(\overline{\sigma_1}(k_1)\sigma_1(k_2)+\overline{\sigma_2}(k_1)\sigma_2(k_2))\,dk_1dk_2$$

in the fermionic one, dk denotes the Haar measure in the p-adic case.

ε-expansions in p-adic models

As was shown earlier the non-Gaussian branch of fixed points can be constructed in the form of ε-expansion in the neighborhood of the Gaussian fixed point, where ε is a deviation of the parameter α from its bifurcation value $3d/2$. We seek this fixed point as the Hamiltonian of the projection of the φ^4-theory in the whole space onto the ball:

$$P(\alpha)(H(\sigma; u, v)) = -\ln\langle \exp\{-H(\sigma + \eta; u, v)\}\rangle_{\mu(d\eta)}, \qquad (17)$$

where σ is defined in the ball Ω, the averaging is taken over the Gaussian field η with binary correlation function

$$\langle \eta(k_1)\eta(k_2)\rangle = \delta(k_1 + k_2)|k_1|^{d-\alpha}(1 - \chi(k_1)) \qquad (18)$$

in the bosonic case and

$$\langle \eta_i(k_1)\overline{\eta_j}(k_2)\rangle = \delta(k_1 + k_2)\delta_{ij}|k_1|^{d-\alpha}(1 - \chi(k_1)) \qquad (19)$$

in the fermionic one. Here

$$H(\sigma; u, v) = uH_2(\sigma) + vH_4(\sigma),$$

ε-expansions in p-adic models

where

$$H_2(\sigma) = \int \delta(k_1 + k_2)\sigma(k_1)\sigma(k_2)\, dk_1 dk_2,$$

$$H_4(\sigma) = \int \delta(k_1 + \ldots + k_4)\sigma(k_1)\ldots\sigma(k_4)\, dk_1 \ldots dk_4$$

in the bosonic case and

$$H_2(\sigma) = \int \delta(k_1 + k_2)(\overline{\sigma_1}(k_1)\sigma_1(k_2) + \overline{\sigma_2}(k_1)\sigma_2(k_2))\, dk_1 dk_2,$$

$$H_4(\sigma) = \int \delta(k_1 + \ldots + k_4)\overline{\sigma_1}(k_1)\sigma_1(k_2)\overline{\sigma_2}(k_3)\sigma_2(k_4)\, dk_1 \ldots dk_4$$

in the fermionic case.

ε-expansions in p-adic models

On expanding of projection Hamiltonian$P(\alpha)H(\sigma; u, v)$ in Feynman diagrams some coefficient functions have a poles in $\varepsilon = \alpha - \frac{3}{2}d$ at zero. In that problem the operation of analytic renormalization $A.R.$ is most natural regularization procedure . Let $H(\sigma; u, v)$ denotes bosonic(or fermionic) Euclidean (or p-adic) Hamiltonian.

Theorem

There are non-trivial formal series $u(\varepsilon)$ and $v(\varepsilon)$ such that the Hamiltonian

$$H_0(\sigma; \alpha) + A.R.P(H(\sigma; u(\varepsilon), v(\varepsilon)) \qquad (20)$$

is invariant under RG-transformation.

One can snow that in the fermionic hierarchical case projection Hamiltonian has the same structure

$$P(\alpha)H(\sigma; u, v) = H(\sigma; r(u, v), g(u, v)),$$

where coefficients $r(u, v)$ and $g(u, v)$ are series of (u, v).

ε-expansions in p-adic models

The action of RG-transformation $R_\lambda(\alpha)$ on the Hamiltonian $H(\sigma; r, g)$ reduces to the transformation of the coupling constants. Particularly, for $\lambda = p^{-1}$

$$R_{p^{-1}}(\alpha)H(\sigma; r, g) = H(\sigma; r', g'),$$

where

$$r' = p^{\alpha-d}\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/p^d}(r+1) - 1\right)$$

$$g' = p^{2\alpha-3d}\left(\frac{(r+1)^2 - g}{(r+1)^2 - g/p^d}\right)^2 g.$$

Fixed point equation

$$R(\alpha)(r, g) = (r, g)$$

has two branches of the non-trivial fixed points.

ε-expansions in p-adic models

The branch, bifurcating from the trivial (gaussian) fixed point at $\alpha = 3d/2$ is called plus-branch and is given by the formula

$$r^+(\alpha) = \frac{p^{d/2} - p^{\alpha - d}}{1 - p^{d/2}},$$

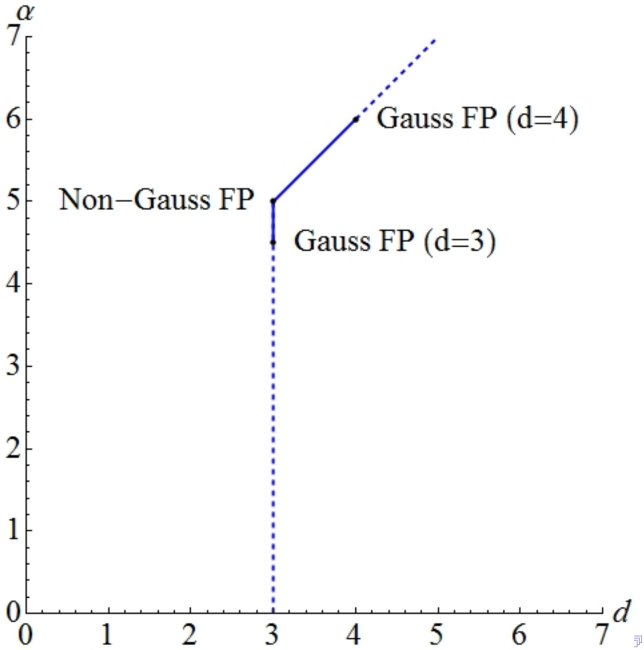
$$g^+(\alpha) = p^d \frac{1 - p^{\alpha - 3d/2}}{1 - p^{\alpha - d/2}} \left(\frac{1 - p^{\alpha - d}}{1 - p^{d/2}} \right)^2 .$$

In physical papers usually α is fixed and is equal to $(d + 2)$. In that case the Gaussian part of the Hamiltonian is given by the Laplace operator. Physicists consider $(4 - d)$ -expansion and try to extrapolate the results of the expansion to the point $d = 3$ (they have a few lower order members of the series with zero convergence radius). If we do the same in the (r, g)-space of the coupling constants of the fermionic hierarchical model, we will see that $d = 4$ is bifurcation value of the parameter d and we can construct $(4 - d)$-expansion from the Gaussian fixed point at the dimension $d = 4$.

ε-expansions in p-adic models

From the explicit formulas for the "plus"-fixed points it follows that $(\alpha - 3/2d)$-expansion and $(4 - d)$-expansion describe the same non-Gaussian fixed point at the dimension $d = 3$. We have some arguments that the same is true in the bosonic hierarchical model [3].

figure 2



References

Missarov M.D. New Variant of the Fermionic Model on the Hierarchical Two-Dimensional Lattice, *p*-Adic Numbers, Ultrametric Analysis and Applications 2020, V.12, pp. 163–170.

Missarov M.D. On Gaussian Self-Similar Random Fields on a 2-Adic Plane , Russian Mathematics, 2020, V.64 , pp. 83 - 87.

Missarov M.D., Stepanov R.G. ϵ-expansions in the Dyson's hierarchical model, Theor. Math. Phys. , 2004, V. 139,268-275.